Structure of the Generalized Veneziano Amplitudes.* I. Radon Transforms

Alfred C. T. Wu
Department of Physics, University of Michigan, Ann Arbor, Michigan
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The (tree-graph) generalized Veneziano amplitudes are shown to be boundary values of a new class of generalized hypergeometric functions having the property that they are Radon transforms of products of *linear* forms.

1. INTRODUCTION

The Veneziano model has attracted considerable interest.[1-12] We address ourselves here to the question of the functional structure of the generalized Veneziano amplitudes.

As is well known, the original Veneziano amplitude for a four-body process was written down as a beta function:

$$V_4(-\alpha_s, -\alpha_t) = \frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t)} = \int_0^1 dx x^{-\alpha_s-1}(1-x)^{-\alpha_t-1}. \tag{1}$$

V_4 can obviously be written as a special case of the Gauss hypergeometric function ${}_2F_1(a,b;c;w)$:

$$V_4(-\alpha_s,-\alpha_t) = \Gamma(-\alpha_s)\,\Gamma(1+\alpha_s) \times {}_2F_1(-\alpha_s, \alpha_s+\alpha_t+1;1;1). \tag{2}$$

Furthermore, Bialas and Pokorski[8] pointed out that the five-particle amplitude as generalized by Bardakci and Ruegg[2] is a special case of the generalized hypergeometric function ${}_3F_2(a,b,c;d,e;w)$ at $w=1$:

$$V_5(-\alpha_{ij}) = \frac{\Gamma(-\alpha_{12})\Gamma(-\alpha_{23})}{\Gamma(-\alpha_{12}-\alpha_{23})}\frac{\Gamma(-\alpha_{34})\Gamma(-\alpha_{45})}{\Gamma(-\alpha_{34}-\alpha_{45})} \times {}_3F_2(\alpha_{15}-\alpha_{23}-\alpha_{34}, -\alpha_{45}, -\alpha_{12}; -\alpha_{34}-\alpha_{45}, -\alpha_{12}-\alpha_{23}; 1). \tag{3}$$

The question immediately arises as to what the corresponding statements for the higher amplitudes are. We wish to make the following observations:

(a) The N-particle Veneziano amplitudes[4-6,9-11] $V_N, N \geq 6$, evidently do not belong to any familiar class of generalized hypergeometric functions. Thus a naive extrapolation that V_6 might be related to some function like ${}_pF_q(a_1\ldots a_p; b_1,\ldots,b_q;w)$ at $w=1$ is obviously false. (b) Radon structure: The class of generalized Veneziano amplitudes [4-6,9-11] corresponding to the so-called tree graphs may be regarded as the boundary values of a new class of generalized hypergeometric functions in several variables. This class of functions satisfies the criterion that they are Radon transforms of products of *linear* forms.[13] On the other hand, this simply property apparently breaks down when one considers the generalized amplitudes (for $N>4$) corresponding to the nonplanar graphs, such as those discussed by Virasoro[14] and Mandelstam.[15]

2. GENERALIZED VENEZIANO AMPLITUDES AS BOUNDARY VALUES OF A CLASS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

We define a class of generalized hypergeometric functions $F^{(n)}(a_i,b_i,c_{ij};w_{ij})$ of $\frac{1}{2}n(n-1)$ variables w_{ij} as follows:

$$F^{(n)}(a_i,b_i,c_{ij},w_{ij}) = \int_0^1\!\cdots\!\int \prod_{k=1}^n dx_k \prod_{k=1}^n [x_k^{\ a_k}(1-x_k)^{b_k}] \times \prod_{i<j}(x_i - w_{ij}x_j)^{c_{ij}} \tag{4}$$

These functions justify the name of generalized hypergeometric functions in the sense of Gel'fand,[13] namely, they are Radon transforms of products of *linear* forms in an $(n+1)$-dimensional space. Obviously, one can write

$$F^{(n)}(a_i,b_i,c_{ij};w_{ij}) = \int\cdots\int \prod_{k=1}^{n+1} dx_k \delta(1-\Sigma x_i)\prod_{k=1}^N (\xi^{(k)},x)^{s_k}, \tag{5}$$

where $N=\frac{1}{2}n(n+3)$ and

$$(\xi^{(k)},x) = \sum_{i=1}^{n+1}\xi_i^{(k)}x_i$$

is a linear form in x with coefficients ξ_i^k. The w's enter through the ξ's.

Special cases of $F^{(n)}$ are easily recognizable, for example,

$$F^{(1)}(a,b) = \Gamma(a+1)\Gamma(-a) \times {}_2F_1(a+1,-(a+b+1);1;1) \tag{6}$$

$$F^{(2)}(a_i,b_i,c_{12};w_{12}) = \frac{\Gamma(a_2+1)\Gamma(b_2+1)}{\Gamma(a_2+b_2+2)}\frac{\Gamma(a_1+c_{12}+1)\Gamma(b_1+1)}{\Gamma(a_1+b_1+c_{12}+2)} \times {}_3F_2(-c_{12}, a_2+1, -(a_1+b_1+c_{12}+1); a_2+b_2+2, -(a_1+c_{12}); w_{12}). \tag{7}$$

As far as the author is aware, $F^{(n)}$ for $n>3$ does not seem to correspond to any known function in the mathematical literature.

It can be easily verified that by simple change of variables, the $(n+3)$-point generalized Veneziano function[4-6,9-11] can be brought to the form of Eq. (4) with all $w_{ij}=1$, namely,

$$V_{n+3} = \text{const}\ F^{(n)}(a_i,b_i,c_{ij};w_{ij}=1), \tag{8}$$

where the parameters [analogs of the α_{ij} in Eq. (3)] entering in V_{n+3} are combinations of the a's, b's, and c's.

To be precise, the suggested change of variables for V_{n+3} consists of letting

$$x_k = \prod_{l=1}^{k} u_l, \qquad k = 1, 2, \ldots, n, \tag{9}$$

where the u's are the usual integration variables [4-6, 9-11] successively associated with the internal lines of a multiperipheral graph. This prescription can be easily checked with the aid of Chan's explicit formulas[4-6] for $n = 3$ and $n = 4$, or with the form written down by Bardakci and Ruegg[11].

In a following paper, we shall study the structure of the generalized Veneziano amplitudes from the point of view of group representations.

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Structure of the Generalized Veneziano Amplitudes.* II. Possible Group Theoretic Content

Alfred C. T. Wu
Department of Physics, University of Michigan, Ann Arbor, Michigan
(Received 31 August 1970)

An attempt is made to generalize a result of Vilenkin that the beta function (being adopted as Veneziano formula for four-particle processes) appears as the kernel (in the integral) of the irreducible representation (in the Mellin-transformed space) of the group of 2×2 unimodular triangular matrices. It is shown that with a modified multiplier in the Gel'fand–Naimark prescription for the representation of the group of $(n + 1) \times (n + 1)$ unimodular triangular matrices, the (tree-graph) generalized Veneziano amplitude for $(n + 3)$-particle processes is recovered by a limiting procedure.

1. INTRODUCTION

In a previous paper,[1] we discussed the functional structure of the $(n + 3)$-point (tree-graph) Veneziano amplitudes viewed as the boundary values of a new class of generalized hypergeometric functions having the property that they are the Radon transforms of products of *linear* forms in an $(n + 1)$-dimensional space.

In this paper, we shall investigate possible group theoretic content of the generalized Veneziano amplitudes.

Many special functions have acquired a new level of respectability as well as a deeper *raison d'être* when it can be shown that a given function occurs naturally in the representation theory of certain groups, even though this connection is in general not one to one. While it is by now more or less a standard textbook exercise[2-5] in going from the representations of known groups to known functions,[6] the converse route (namely, given the function, finding the group) is obviously much more hazardous. Nevertheless, we venture to point out that, in the mathematical literature, certain results exist for the connection between the representations of the group $SL(2, R)$ and the Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ and, in particular, the special case of triangular matrices of $SL(2, R)$ yields the beta function as the kernel.[7]

The apparent interest in the Veneziano model[8] perhaps justifies an attempt to generalize the Vilenkin result for the beta function.

A straightforward application of the Gel'fand–Naimark scheme[9] for the irreducible representations of $(n + 1) \times (n + 1)$ unimodular triangular matrices will yield in general a class of functions different from the class of functions $F^{(n)}$ discussed in Paper I. The differences generally are twofold. (i) In one aspect, the multiplier in the Gel'fand–Naimark scheme, which consists of the products of n principal minors[10] $\Delta_{p\, p+1 \cdots n+1}$ ($p = 2, \ldots, n + 1$), is found to be insufficient in generating all the desired tree-graph links in the Veneziano formula. One can remedy this if one modifies the